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A Pioneer in Anaclastics

Ibn Sahl on Burning Mirrors and Lenses

By Roshdi Rashed*

THE GEOMETRICAL STUDY OF LENSES was essential for the development of optics in the late sixteenth and early seventeenth centuries.[1] This study, which historians have seen as a turning point in the history of optics, was designated as either *anaclastics* or *dioptrics*. In writing the history of this chapter, it is common practice to give prominence to Kepler, some of Mersenne's circle, Willebrord Snellius, and Descartes. Furthermore, the perceived modernity of the optics of this period is frequently explained, partially at least, by external reasons: a very modest technical advance in the construction of optical instruments.

A reading of the eleventh-century *Book of Optics* (*Kitāb al-Manāẓir*) by Ibn al-Haytham, however, whether in Arabic or in Latin translation, should have suggested to historians that research on anaclastics started well before the late sixteenth century. In Book 7 there is a study of the spherical diopter and the spherical lens.[2] Furthermore, Ibn al-Haytham devoted an entire memoir to the burning sphere, whose scientific and historical importance is unanimously recognized. This work contains an examination of double refraction in the sphere and of related problems. Three centuries later, Kamāl al-Dīn al-Fārisī wrote a commentary on the work and used it for the first correct explanation of the rainbow.[3]

Until now, however, historians have not constructed a clear picture of Ibn al-Haytham's research on lenses and of anaclastics in general. For example,

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[1] This topic is encountered in numerous works on the history of optics—in particular, in attempts to understand Descartes's contribution to the discipline. On this subject see Gaston Milhaud, *Descartes savant* (Paris: Félix Alcan, 1921), Ch. 5; and the commentary by Cornelis de Waard to his edition of *Correspondance du P. Marin Mersenne, religieux minime,* 7 vols. (Paris: Presses Universitaires de France, 1933–1962), Vol. I: *1617–1627*.

[2] The texts relating to lenses in Book 7 of Ibn al-Haytham's work have been edited, translated into French, and analyzed in Roshdi Rashed, *Géométrie et dioptrique au Xe siècle: Ibn Sahl, al-Qūhī, et Ibn al-Haytham* (Collection Sciences et Philosophie Arabes, Textes et Études) (Paris: Les Belles Lettres, forthcoming).

[3] Al-Fārisī's commentary, which more or less reproduces Ibn al-Haytham's text, has been translated freely into German by Eilhard Wiedemann and was recently reexamined by Matthias Schramm: see Wiedemann, "Beiträge zur Geschichte der Naturwissenschaften, XIX: Über die Brechung des Lichtes in Kugeln nach Ibn al-Haytham und Kamāl al-Dīn al-Fārisī," *Sitzungsberichte der physikalische-medizinischen Sozietät in Erlangen,* 1910, *13*:15–57; and Schramm, "Steps towards the Idea of Function: A Comparison between Eastern and Western Science in the Middle Ages," *History of Science,* 1965, *4*:70–103. I have edited, translated into French, and analyzed al-Fārisī's commentary, including Ibn al-Haytham's original text: see Rashed, *Géométrie et dioptrique*.

Mustafa Nazif's masterful study of Ibn al-Haytham's investigations in optics is still unsurpassed, and his analyses have frequently been borrowed. But even Nazif, having ascertained that Ibn al-Haytham dealt with the study of lenses, was incapable of extracting the exact meaning of his research and concluded with some remarks that leave us perplexed.[4] This weakness can be ascribed to several causes: incomplete knowledge of Arabic optics before Ibn al-Haytham; a dogma springing from this ignorance, according to which no one before him made effective use of Ptolemy's *Optics;*[5] and an underestimation of how much works on burning mirrors and instruments have contributed to the history of optics. The only predecessor of Ibn al-Haytham of any importance to be mentioned by historians is the tenth-century philosopher and scholar al-Kindī, but for his *De aspectibus* rather than for his *On Burning Mirrors.* When historians analyze Ibn al-Haytham's dioptrics, they refer only to Ptolemy. The result is that Ibn al-Haytham appears in their work as a singular occurrence in the late tenth and early eleventh centuries, preceded by a vacuum reaching back to Ptolemy and followed by another vacuum up to al-Fārisī.

In this article we shall see that this picture of history is inaccurate. I shall show that (1) Ibn al-Haytham was not the first to have effectively used Ptolemy's *Optics,* and consequently the dogma of a vacuum is unfounded; (2) al-Kindī was not the only significant figure in the history of Arabic optics before Ibn al-Haytham—rather, there existed a tradition of research in this field that included names just as prestigious; and (3) it is vital to take into account research on burning mirrors and instruments in order to understand not only the history of catoptrics at that time but also dioptrics.

Some years ago, I discovered and began to reconstruct a treatise on burning instruments written around 984 by a mathematician connected with the court of Baghdad, Abū Saᶜd al-ᶜAlāᵓ Ibn Sahl, whose work was known to Ibn al-Haytham and was even sometimes copied in his own hand (see Fig. 1).[6] Ibn Sahl not only knew Ptolemy's *Optics* but, as we shall see, went further than he did in the study of refraction.[7] This treatise, *On the Burning Instruments,* makes Ibn Sahl the first mathematician known to have studied lenses and shows that in the

[4] Mustafa Nazif, *Al-Ḥasan ibn al-Haytham, buḥūthuhu wa kushūfuhu al-Baṣariyya,* 2 vols. (Cairo: Univ. Cairo, Faculty of Engineering, 1943), Vol. II, sect. 227.

[5] A. I. Sabra, e.g., wrote in 1987, "It is remarkable that no one in late antiquity or in the Islamic world seems to have made any effective use of Ptolemy's *Optics* until Ibn al-Haytham": *Dictionary of the Middle Ages* (New York: Scribners, 1982–), Vol. IX (1987), p. 245.

[6] The treatise is soon to be available in Rashed, *Géométrie et dioptrique* (cit. n. 2). Ibn Sahl was at the full height of his activity in the second half of the tenth century; this treatise was composed between 983 and 985: on these dates and for a biography of Ibn Sahl see *ibid.* In his *Discourse on Light* Ibn al-Haytham referred explicitly to Ibn Sahl and recalled some of his ideas on the transparency of media and refraction; see Roshdi Rashed, "Le *Discours de la lumière* d'Ibn al-Haytham: Traduction française critique," *Revue d'Histoire des Sciences,* 1968, *21*:197–224. Ibn al-Haytham also copied Ibn Sahl's opuscule entitled *Proof That the Celestial Sphere Is Not Completely Transparent*; see Rashed, *Géométrie et dioptrique.*

[7] For the last century it could have been known, on the strength of Ibn Sahl's *Proof That the Celestial Sphere . . . ,* that he undertook an examination of Book V of Ptolemy's *Optics.* In the beginning of this work we read of "a proof deduced by Ibn Sahl when he examined Ptolemy's book on optics, which he wanted to incorporate in the overall examination of Book V of this work." This opuscule has not been studied, any more than Ibn Sahl's other optical and mathematical works; Eilhard Wiedemann simply mentioned its name, which he found in a catalogue of manuscripts in the St. Petersburg library. See Wiedemann, "Bemerkung zu dem Aufsatz von Herrn Dr. J. Baarmann: Abhandlung über das Licht von Ibn al-Haitham," *Zeitschrift der Deutschen Morgenländischen Gesellschaft,* 1884, *38*:145–148.

first half of the tenth century catoptricians were actively working on refraction. When Ibn Sahl had completed his examination of burning mirrors, both parabolic and ellipsoidal, he considered hyperbolic plano-convex lenses and hyperbolic biconvex lenses. Moreover, he succeeded in stating Snellius's law long before Snellius himself, and he studied the mechanical drawing of the three conic curves.

I am not unaware that these results may come as a surprise. They invite us to reexamine the history of the beginnings of anaclastics and the genesis of its concepts in a new light. Let us remember that Ibn Sahl's geometrical study of lenses runs parallel with that of burning mirrors and that, at its birth, anaclastics was the daughter of catoptrics and unconcerned with the study of conditions of vision.

I. IBN SAHL'S TREATISE ON BURNING INSTRUMENTS

It has long been known that Ibn Sahl wrote on burning mirrors: libraries in both Damascus and Tehran contain a manuscript bearing this title. It was thought, on the basis of catalogue information alone, that these were two copies of one and the same manuscript.[8] This is, however, not the case: not only do the manuscripts contain different texts, but they have been found to contain no passages in common. In fact, each manuscript reproduces a separate section of Ibn Sahl's original work: that is, the Damascus manuscript contains a long fragment missing from the Tehran copy. The latter is much more substantial, but it is seriously damaged and the sheets are out of order.

My first task was to discern the latent structure of the extant Tehran manuscript and find out how it was organized. This was only possible once I had understood the underlying plan of the whole treatise itself. It was then easy to rearrange the Tehran manuscript, specify its missing parts, and prove that the Damascus fragment was indeed one of them. I then determined where to insert it and filled in some other gaps. Only then was it possible to verify the overall plan of Ibn Sahl's treatise, provide a definitive reconstruction of what has survived, establish and translate the text, and comment on it as well as on all Ibn Sahl's extant works on optics and mathematics.

It is now clear that we possess the major part of Ibn Sahl's treatise on burning instruments, and the absence of certain fragments in no way hinders comprehension; it is quite easy to surmise what their contents were. Providentially, the most significant section on lenses has survived in full.

First, let us recall the problem posed by Ibn Sahl and the various steps necessary for its solution. By thus exposing the underlying plan of his treatise, we can show how the treatise is organized, designate precisely what is missing, and undertake its final reconstruction.

The problem Ibn Sahl tackled may be stated as follows: To burn at a given point *A*, using a distant or near luminous source, by reflection or refraction. To solve this problem, it is necessary to examine, on the one hand, (a) reflection and

[8] This error occurs in Fuat Sezgin's bibliographical work *Geschichte des arabischen Schrifttums-Astronomie* (Leiden: Brill, 1978), Vol. VI, p. 233. The two manuscripts are Damascus, al-Ẓāhirīya MS 4871, 3 fols. (hereafter **D**); and Tehran, Millī MS 867, 51 fols. (hereafter **T**). The first is entitled *Fī al-ᵓāla al-muḥriqa* (On the burning instrument); the second has no title, apart from the one added in another hand on fol. 1r: *Kitāb al-ḥarrāqāt ᶜamilahu Abū Saᶜd al-ᶜAlāᵓ Ibn Sahl* (The book of burners composed by Abū Saᶜd al-ᶜAlāᵓ Ibn Sahl).

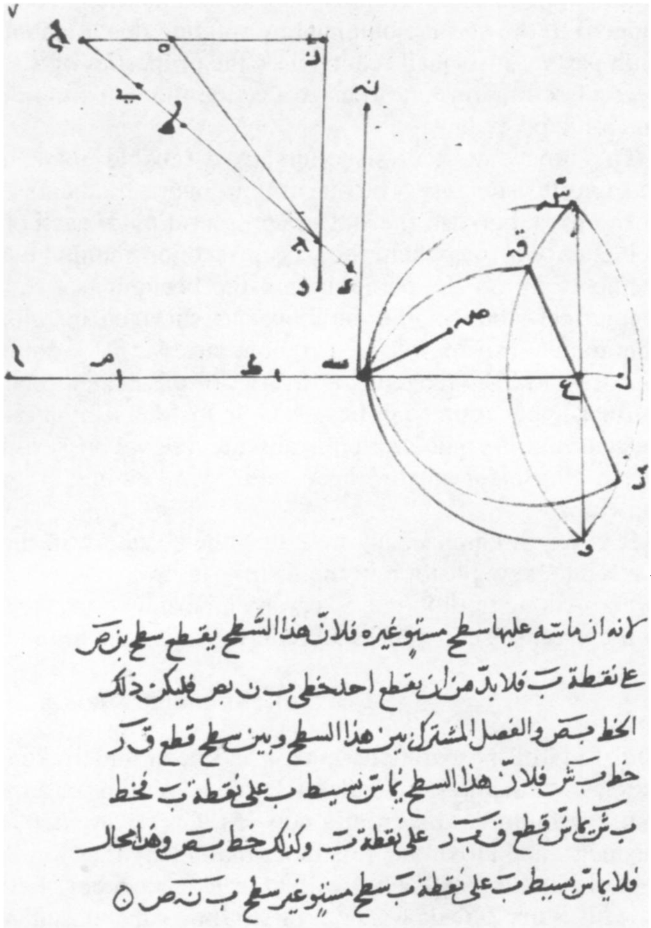


Figure 1. *The plano-convex lens from Ibn Sahl's treatise* On the Burning Instruments; *compare Figures 11, 12. From Millī MS 867, folio 7r; courtesy of Millī Library, Tehran.*

(b) refraction; and on the other hand, (c) the case where rays can be considered parallel and (d) the case where rays come from a point at a finite distance. By combination, we obtain the following: (a) and (c), which indicate a parabolic mirror; (a) and (d), an ellipsoidal mirror; (b) and (c), a plano-convex lens; and last (b) and (d), a biconvex lens. Therefore, following the introduction, Ibn Sahl's treatise should comprise the four chapters indicated.

We should also remark that Ibn Sahl intended to construct these burning instruments. He thus could not limit himself to a theoretical study of each curve but, like others studying burning mirrors, needed to explain how to draw the curves; consequently, each chapter should comprise two parts—one theoretical and one practical. And, in fact, the parts of his treatise that have survived intact verify these assumptions. For instance, the chapter on the hyperbola is divided into two sections: a study of the curve as a conic section, and the continuous drawing of the curve. In the first part, Ibn Sahl defines the hyperbola by the vertex, the axis, and the *latus rectum;* he examines the tangent from the bifocal property; and he then goes on to study the hyperboloid and the tangent plane, whose uniqueness he demonstrates. In the second part, he proceeds with the continuous drawing of the arc of a curve that is none other than a hyperbolic arc, although he does not identify it as such, and then takes up the study of the plane tangent to the surface obtained by rotating this arc about a fixed straight line. In both parts, as we shall see, he uses the properties of the tangent to rediscover the laws of refraction, and thus to deduce the construction of a plano-convex lens and a biconvex lens.

The preceding analysis offers us a reliable guide to reconstructing *On the Burning Instruments*. The diagram in Figure 2 indicates not only the organization of the work but also the state of preservation of each of its sections.

It is easy to see where the missing section should be inserted: between the end of the study on the parabola and the beginning of the one on the ellipse. The theoretical study of the parabola has survived in full, but of the study of the continuous drawing of the parabolic arc we have only a fragment: we lack the discussion of the tangent to this arc, of the tangent plane to the paraboloid, and of their application to optics. And as for the section on the ellipse, the study of this curve as a conic section is missing, but we possess an almost complete study of the ellipsoidal mirror engendered by an elliptic arc obtained by a continuous drawing.

It is worth emphasizing here that the structure of the treatise itself illustrates Ibn Sahl's new position in the history of optics: although his work continues the Greco-Arabic tradition of research on burning mirrors, his introduction of refraction and lenses constitutes a break with that tradition as well.

II. THE PARABOLIC MIRROR

The study of the parabolic mirror had been undertaken long before Ibn Sahl by Diocles, Anthemius of Tralles, "Dtrūms" (author of a treatise on burning mirrors translated into Arabic from a now-lost Greek original), the author of the Bobbio fragment, and al-Kindī.[9] It is more than likely that Ibn Sahl was familiar not only with al-Kindī's treatise but with at least a fragment of Anthemius's work, as well as with other Greek writers. First, Ibn Sahl himself affirms that he consulted some Greek texts translated into Arabic. He claims, moreover, that Hellenistic writers dealt only with burning by reflection, and that he was the first to study burning by refraction. That he studied other Greek authors is also confirmed by a systematic comparison of his study of the parabolic mirror with theirs. Ibn Sahl, for example, did not adopt Diocles' method, but approximated that applied by "Dtrūms." He also mentions the legend according to which Archimedes set fire to the Roman flotilla using burning mirrors. This suggests that he read the work of Anthemius of Tralles, who relates the legend in a fragment on burning mirrors that was translated into Arabic at least a century before, for it was consulted by

[9] See *Dioclès, Anthémius de Tralles, Didyme . . . : Sur les miroirs ardents*, trans. and ed. Roshdi Rashed (Paris: Les Belles Lettres, forthcoming); or see Johan Ludvig Heiberg, *Mathematici Graeci Minores* (*Kongelige Danske Videnskabernes Selskab: Historisk-filologiske Meddelelser,* 13.3) (Copenhagen: Hast, 1927), pp. 77–92; and Wilbur Knorr, "The Geometry of Burning-Mirrors in Antiquity," *Isis,* 1983, *74*:53–73). See also al-Kindī, *Kitāb fī al-Shuᶜāᶜāt,* Patna, India, Khuda Bakhsh MS 2048. For a description of the text ascribed to "Dtrūms," British Library MS 7473, see *Dioclès, Anthémius*, ed. Rashed; and William Cureton, *Catalogus codicum manuscriptorum orientalium qui in Museo Britannico asservantur,* 3 vols., Vol. II (London, 1846–1871), p. 205: "Codex bombycinus in quarto maiori, ff. 198, extractus A.H. 639, A.D. 1242." The Arabic spelling of the name, as well as the presence of a fragment attributed to Didymus, precludes the first name's being an alteration of the second.

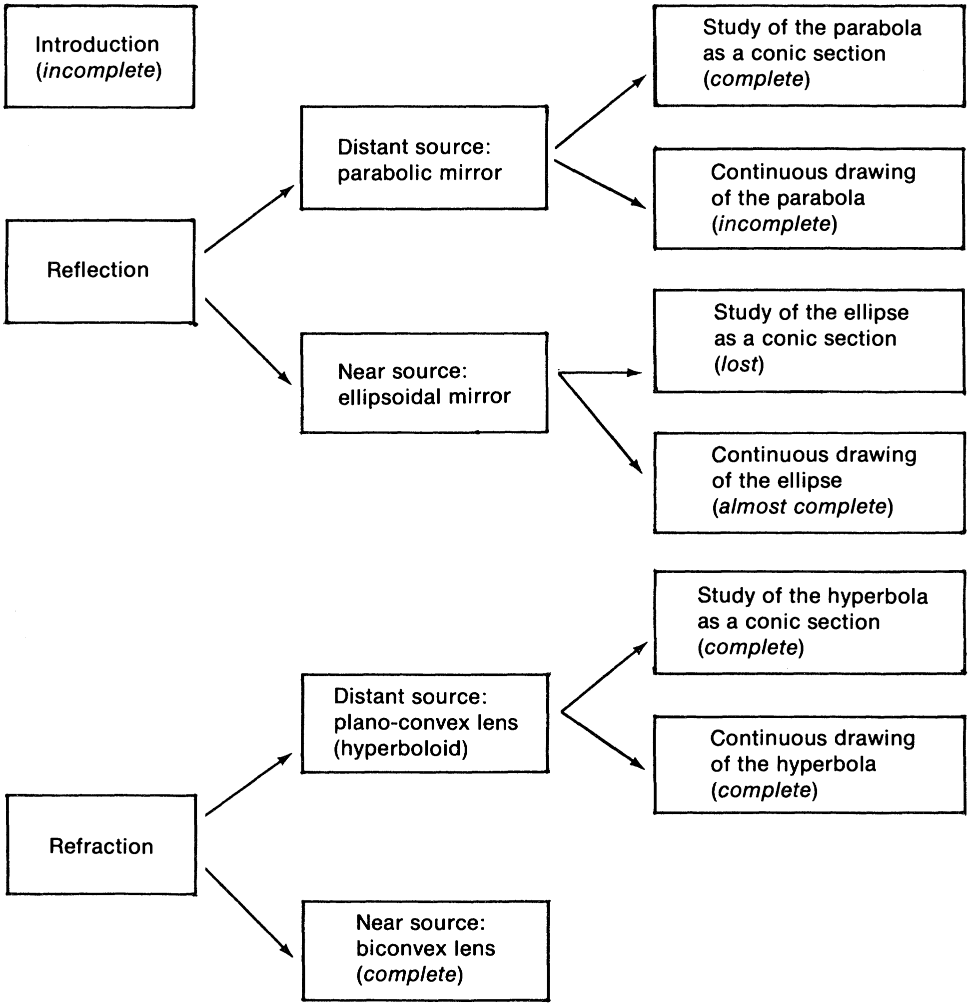


Figure 2. *The structure of* On the Burning Instruments.

al-Kindī. This conjecture is supported by the fact that Anthemius's text is the only one on burning mirrors translated from Greek into Arabic that examines the ellipsoidal mirror, and it is precisely the study of this mirror that Ibn Sahl resumes.[10] Ibn Sahl's study differs from these earlier texts and thus merits close examination.

The parabola as a conic section

Ibn Sahl used the following steps when determining how to burn at a given distance using sunlight.[11]

[10] For Ibn Sahl's claims see T, fol. 1v. On Ibn Sahl's study of the parabolic mirror see *Dioclès, Anthémius*, ed. Rashed; see also the end of this subsection. The Arabic version of Anthemius, previously considered lost, has now been found, and an edition with translation is to be published in *Dioclès, Anthémius*.

[11] D, fol. 81r.

Let AB be this distance and AC the direction of solar rays (Fig. 3). Suppose AC perpendicular to AB, and set $AC = AB/2$. Draw $CD \perp AC$, such that $CD \cdot AC = AB^2$. The parabola of vertex C, with axis AC and *latus rectum* equal to the distance CD, passes through B.

Consider arc BE on this parabola in the opposite direction to C. If we rotate arc BE about the fixed straight line AC, B describes a circular arc BF, and E describes a circular arc EG. We thus delimit a portion of the paraboloid $EBFG$, written (BG).

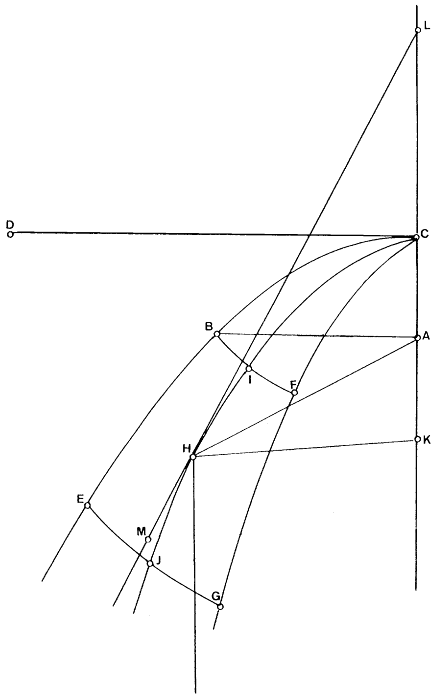


Figure 3. *This and the following line figures reproduce those of Ibn Sahl, with corrections as necessary.*

PROPOSITION. *Rays parallel to* AC *falling on surface* (BG), *supposed reflective, are reflected to point* A.

To demonstrate this proposition, Ibn Sahl starts by discussing the tangent plane at point H and the uniqueness of the tangent plane.

> Let H be a point on (BG); plane ACH cuts (BG) along arc IJ, a parabolic arc equal to arc BE. Let K be the orthogonal projection of H on AC, and let L be a point on AC, such that $CL = CK$; then line LH is tangent to arc IJ. The plane that passes through line LH and is perpendicular to plane AHC is tangent to surface (BG) at H.

By *reductio ad absurdum* Ibn Sahl shows that this plane cannot cut surface (BG) at point H, and then that the plane tangent to H is unique.

In stage two Ibn Sahl discusses the reflection of a ray of light parallel to the axis.

Let HX be the ray falling at H, and M a point on the prolongation of LH; we must show that $\sphericalangle MHX = \sphericalangle AHL$.

We have

$$CD \cdot AC = AB^2 = 4AC^2, \text{ hence } CD = 4AC.$$

On the other hand, H is on the paraboloid, hence

$$HK^2 = CD \cdot KC + 4AC \cdot KC.$$

We deduce from the above that

$$\begin{aligned} AH^2 &= AK^2 + 4AC \cdot KC = AK^2 + 4AC^2 + 4AC \cdot AK \\ &= (AK + 2AC)^2 = AL^2, \end{aligned}$$

and consequently

$$\sphericalangle AHL = \sphericalangle ALH.$$

But as $HX//AL$, we have $\sphericalangle ALH = \sphericalangle MHX$, hence $\sphericalangle MHX = \sphericalangle AHL$. The ray XH falling at H is reflected to point A.

Ibn Sahl then considers the case where AC is not perpendicular to AB.

Figure 4

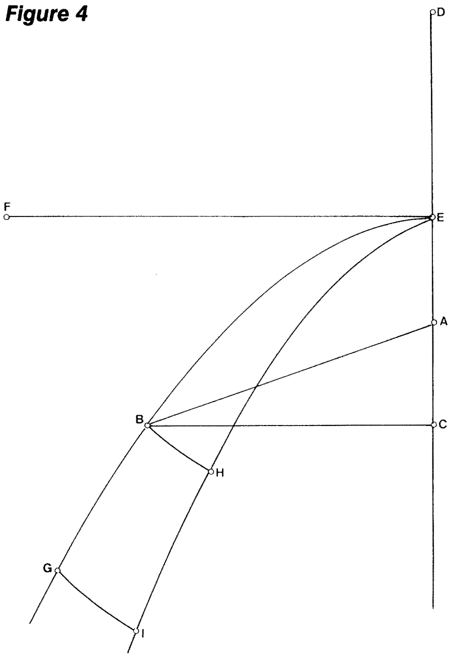
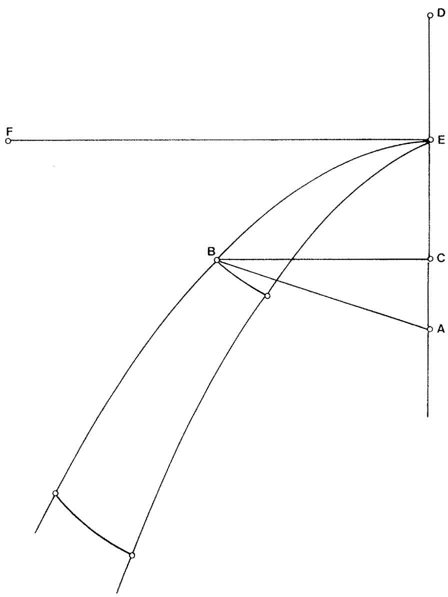


Figure 5



Draw the perpendicular to AC from B; let C be the foot of this line; and draw a length $AD = AB$ on line AC. There are two possibilities: C and D on either side of A (Fig. 4), and C and D on the same side of A (Fig. 5). Let E be the midpoint of CD, and EF the perpendicular to CD, such that $EF \cdot CE = BC^2$. The parabola of vertex E, with axis AE and *latus rectum* equal to the distance EF, therefore passes through B.

We consider arc BG on this parabola, and the portion of the paraboloid (BI) obtained by rotation about AC. If this portion of the paraboloid is reflective, any ray parallel to AC, falling at a point on this surface, is reflected to point A.

To demonstrate the proposition in both these cases, Ibn Sahl wants to arrive at the preceding case. It therefore suffices to show that A is the focus of the parabola—that is, that $EA = \frac{1}{4}EF$. He proceeds as follows:

We have

$$EF \cdot CE = BC^2, \text{ and } AB^2 = AC^2 + BC^2 = AC^2 + EF \cdot CE.$$

But if we consider both possibilities, we can have

$$AD = 2EC - AC, \text{ and } AE = EC - AC \text{ (Fig. 3)},$$

or

$$AD = 2EC + AC, \text{ and } AE = EC + AC \text{ (Fig. 4)}.$$

We therefore have

$$\begin{aligned} AD^2 &= AC^2 + 4EC^2 \pm 4EC \cdot AC = AC^2 + 4EC(EC \pm AC) \\ &= AC^2 + 4\mathrm{EC} \cdot \mathrm{AE}. \end{aligned}$$

We deduce that $EC \cdot EF = 4EC \cdot AE$, and hence $EF = 4AE$.

Thus point A lies at a distance from vertex E of the parabola equal to a quarter of the *latus rectum*. Therefore, as in the first case, any ray parallel to the axis falling on mirror (BI) is reflected to point A.

Ibn Sahl has thus shown that in all three cases—$\sphericalangle BAC = \pi/2$, $\sphericalangle BAC < \pi/2$, and $\sphericalangle BAC > \pi/2$—rays parallel to the axis are reflected to point A on the axis whose distance from the vertex is a quarter of the *latus rectum*.

We note that in his demonstration Ibn Sahl resorts to the fundamental relation, the *symptōma,* of the parabola, and to the property of the vertex of a parabola, which is to be the midpoint of the subtangent, in considering the three cases. As mentioned earlier, this approach differs from that of Diocles, who stated the same proposition by resorting to the fact that the subnormal is equal to the parameter—without using the *symptōma*. In the Greek text translated into Arabic and attributed to "Dtrūms," on the other hand, the author used the same auxiliary propositions as Ibn Sahl to demonstrate the same main property, but his starting point was different: instead of starting, like Ibn Sahl, with the focus to establish the equality of the angles, he started with this equality to determine the focus. Ibn Sahl's approach bears the closest resemblance to that followed in the Bobbio fragment, but there is no indication that the latter text was translated into Arabic. Moreover, it appears that Ibn Sahl was the first, so far as we know, to examine the problem of the uniqueness of the tangent plane.

If we compare Ibn Sahl's study with that of Ibn al-Haytham's *On the Burning Parabolic Mirror,* we encounter the same proposition and the same demonstration, even if the latter's exposition is somewhat improved and proceeds by "analysis" and "synthesis."[12]

[12] The points made in n. 6 also show that Ibn al-Haytham was acquainted with Ibn Sahl's treatise and followed the same method.

Drawing the parabola

Ibn Sahl then proceeds with the continuous drawing of the parabola, using the focus and the directrix.[13]

Take a fixed point A, a fixed straight line DF, and a length $DE = l$ produced on the perpendicular at DF. Let AC be the perpendicular drawn to DF from A; A and E lie on either side of DF, and $DE > AC$ (Fig. 6).

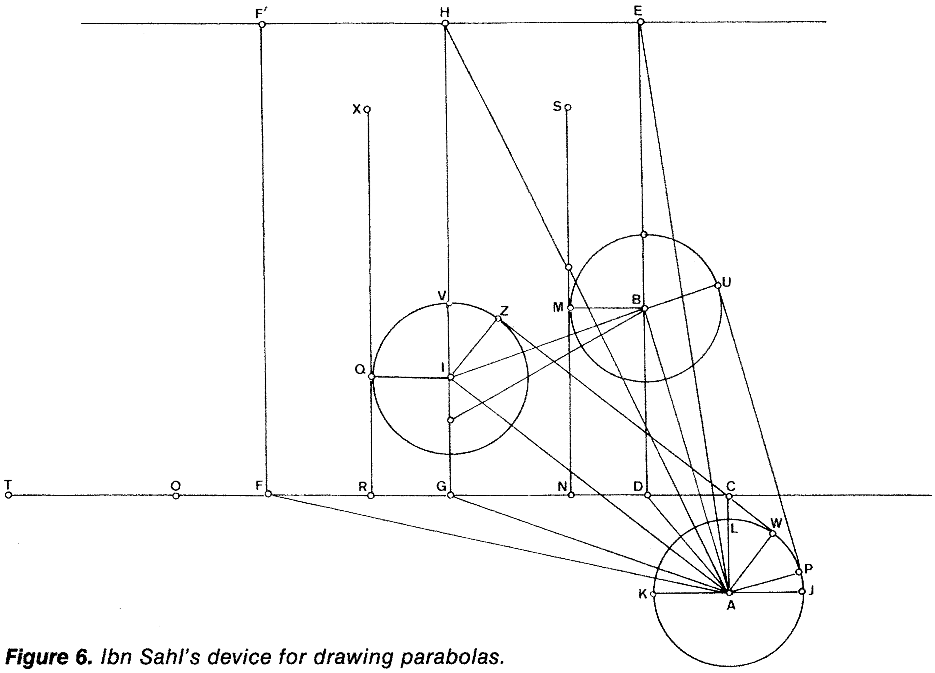


Figure 6. *Ibn Sahl's device for drawing parabolas.*

Ibn Sahl explains the construction of three points that belong to the parabola of focus A and whose directrix is line EH parallel to DF. (Note that at this point he does not name the parabola.)

Situate points F and B on DE, and I on GH, a segment perpendicular to DF, such that their locations satisfy the equalities

$$AF = l,\ BE = BA, \text{ and } IH = IA,$$

and consequently

$$BD + BA = IG + IA = FA = l. \tag{1}$$

Points C, D, G, and F follow one another in this order on DF. We prove by *reductio ad absurdum* that $AI > AB$.

Draw a semicircle with center A and diameter JK, with $JK \leqslant AB$, and two circles with centers B and I, respectively, all with the same radius. The hypothesis $JK \leqslant AB$ implies that $JK < AI$; therefore circles (A) and (B) on the one hand, and (A) and (I) on the other, do not intersect each other. Construct PU tangent to both (A) and (B), and MN tangent to (B) and perpendicular to DF. We deduce that

[13] T, fols. 14–17.

$$PU = AB, \; MN = BD, \text{ and } \widehat{PK} = \widehat{UM}.$$

The outline *JPUMN* therefore has a length s_1.

Call the semiperimeter of one of the circles *p;* we have

$$s_1 = \widehat{JP} + PU + \widehat{UM} + MN = l + p.$$

Using the same procedure, we associate outline *JWZQR* with circle (*I*), and we have

$$s_2 = \widehat{JW} + WZ + \widehat{ZQ} + QR = l + p.$$

The procedure used by Ibn Sahl to arrive at a continuous drawing is deduced from the equality $s_1 = s_2$, which results from equality (1).

> Take a rigid set-square, and let one of its right-angle sides, *NO,* slide along *DF,* while the other side, *NS,* is applied to *NM; NS* is chosen greater than *NM*. Point *A* is fixed, as is semicircle (*A*). Circle (*B*) is mobile and in contact with a belt of length $l + p$, one of whose ends is attached at *J* on semicircle (*A*) and the other at *N* on the set-square. We suppose the belt inelastic.
>
> If we push on circle (*B*) while keeping the belt taut, with circle (*B*) remaining in contact with side *NS* of the set-square, the latter slides along line *DF,* which functions as a rail. A stylus placed at point *B* draws a parabolic arc *BI*. Note that point *B* can be displaced to either side of its starting position: as far as the vertex of the parabola on one side, and until the mobile circle (*B*) is tangent to line *DF* on the other side.

The last part of this study on the continuous drawing of the parabola, unfortunately lost, must have included—as the other chapters lead us to suppose—a study of the tangent to a point of the arc *BI* just constructed, the tangent plane to the surface engendered by this arc, and lastly, the reflection of a light ray on this surface. This lost study also would have aimed at verifying that the mirror so constructed—by the focus and the directrix—is indeed a parabolic mirror. In the tenth century, at least for Ibn Sahl, the focus-directrix property did not yet *define* the parabola as a geometrical locus of points.[14]

III. THE ELLIPSOIDAL MIRROR

The only known work on the ellipsoidal mirror prior to that of Ibn Sahl is a preliminary study by Anthemius of Tralles, in which he used the bifocal property of the ellipse and affirmed, without further explanation, that by virtue of the laws of reflection, a ray from one focus is reflected toward the second focus. Anthemius also invoked the procedure known as the "gardener's method" for the continuous drawing of the ellipse.[15] It seems highly likely, as I have observed, that Ibn Sahl was acquainted with the Arabic version of Anthemius's work. However, as noted earlier, the section of his text on the study of the ellipse as a conic section is lost; only the passage on the continuous drawing of the ellipse has survived.

[14] See Rashed, *Géométrie et dioptrique* (cit. n. 2).

[15] Heiberg, *Mathematici Graeci Minores* (cit. n. 9); and Thomas L. Heath, "The Fragment of Anthemius on Burning Mirrors and the 'Fragmentum mathematicum Bobiense,' " *Bibliotheca Mathematica,* 3rd Ser., 1906/7, *7*:225–233.

Drawing the ellipse

To draw an elliptic arc, Ibn Sahl starts with three nonaligned points, A, B, and C, such that $AB < AC < BC$ (Fig. 7).[16]

> Place point D on line CB, such that $CB + BA = CD = l$. On a circle with center C and radius l place point E such that $\sphericalangle ACB < \sphericalangle ACE \leqslant \sphericalangle CAB$, B and E being on the same side of the line CA; and place point F on segment CE such that $AF = EF$. We therefore have $FA + FC = l$. Hence, points B and F belong to the ellipse of focuses A and C with circle (C, l) as a directrix circle.

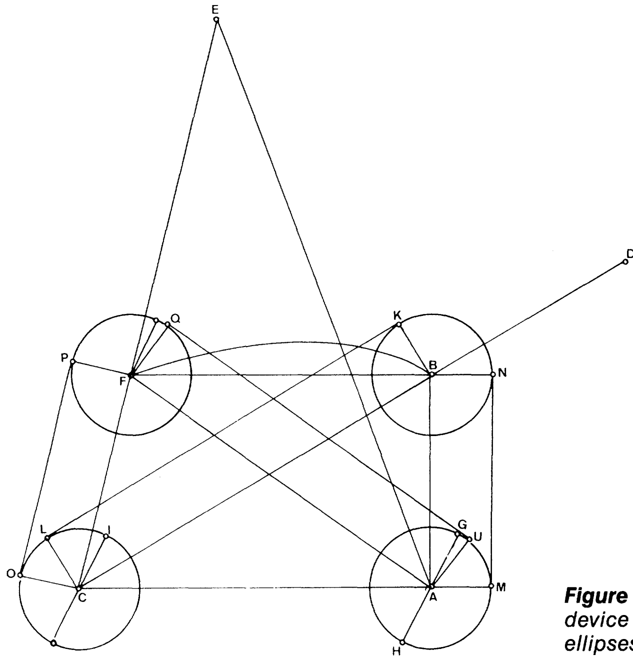


Figure 7. *Ibn Sahl's device for drawing ellipses.*

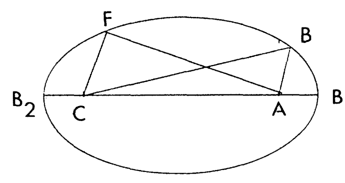
Just as in the case of the parabola, Ibn Sahl does not name the ellipse when he sets out the method for the continuous drawing of arc BF thus defined.

> From the hypotheses made for the construction of F it follows that $AF > AB$, an inequality that can be proved by *reductio ad absurdum*, and consequently $CF < CB$. It also follows that $CF \geqslant AB$.[17]
>
> Draw two parallel, equal segments, GH and IJ, with centers A and C, respectively—$IJ = GH < AB$—and draw circles (A), (C), (B), and (F), of radius $\frac{1}{2}GH$, with centers A, C, B, and F, respectively. The hypothesis $GH < AB$ entails that these circles taken in pairs do not intersect each other.
>
> If MN is a tangent common to (A) and (B), and KL a tangent common to (B) and (C), we have

[16] T, fols. 13, 2.

[17] Let there be an ellipse of focuses A and C, with major axis B_1B_2. If B describes arc B_1FB_2, distance AB increases from AB_1 to AB_2, and consequently CB decreases:

$\sphericalangle ACF > \sphericalangle ACB \rightarrow AF > AB$
$\sphericalangle ACF \leqslant \sphericalangle CAB \rightarrow CF \geqslant AB$



$$MN = AB \text{ and } KL = BC, \text{ whence } MN + KL = l.$$

On the other hand, $AM//BN$, $BK//CL$, $AH//CJ$ implies $\widehat{HM} + \widehat{NK} + \widehat{LJ} = p$, the perimeter of one of the circles. The outline $HMNKLJ$, of length s_1, is associated with circle (B):

$$s_1 = \widehat{HM} + MN + \widehat{NK} + KL + \widehat{LJ} = l + p.$$

Similarly, draw UQ tangent to both (A) and (F), and PO tangent to (F) and (C). With circle F we associate the outline $HUQPOJ$, whose length is

$$s_2 = \widehat{HU} + UQ + \widehat{QP} + PO + \widehat{OJ}.$$

We have, as above,

$$UQ + PO = AF + FC = l, \text{ and } \widehat{HU} + \widehat{PQ} + \widehat{OJ} = p;$$

therefore

$$s_2 = l + p = s_1.$$

To draw the curve Ibn Sahl imagined a device consisting of three circles with the same radius, functioning as pulleys, and a belt with a constant length $l + p$. Two of the circles are fixed, with centers A and C; the third, with center B, is mobile. The belt, with one end fixed to circle (A) at H, and the other to circle (C) at J, revolves around pulley B (Fig. 8). If we push on circle (B) while keeping the belt taut, center B describes an elliptic arc BF.

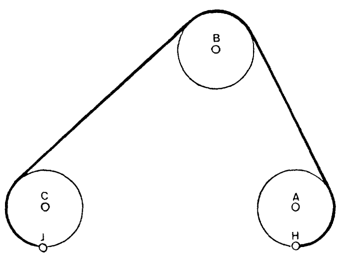


Figure 8. *Another device for drawing ellipses.*

Ellipsoidal reflection

While the theoretical study of the ellipse as a conic section has been lost, that of the reflection on an ellipsoidal mirror has been preserved in its entirety, unlike the case for the parabola. Starting with the bifocal property, Ibn Sahl wants to show that rays falling on one of two focuses are reflected to the other focus and may therefore burn.[18]

By rotating arc BF about line AC we engender a surface (BX), B describing a circular arc BG, and F, a circular arc FX. It can be shown that light rays from C are reflected to A.

Let T be any point on arc BF; it is associated with a circle (T) and with an outline of length s. Circle (T) corresponds to a position of circle (B), and therefore we have $s = s_1$; we deduce that $TA + TC = BA + BC$ (Fig. 9).

Let I' be any point on (BX); plane $AI'C$ cuts (BX) along arc B_aO', a position of arc FB; we therefore have $I'A + I'C = BA + BC$ (Fig. 10). If we

[18] T, fols. 2–5.

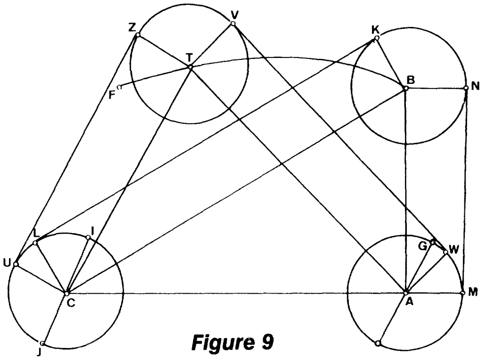


Figure 9

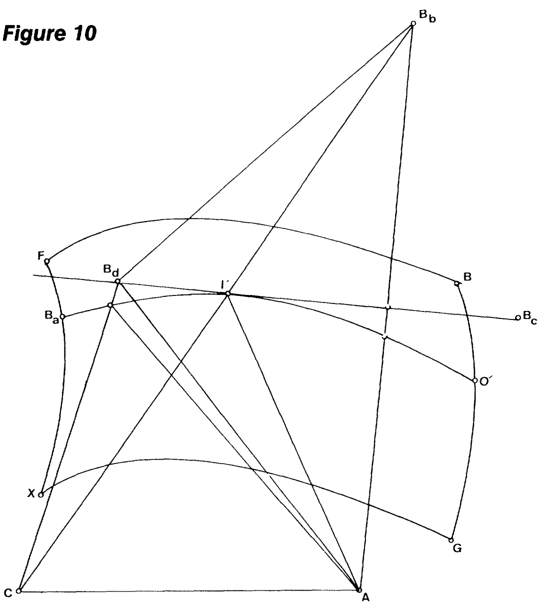
prolong CI' by length $I'B_b = I'A$, the bisector $B_cI'B_d$ of $\sphericalangle AI'B_b$ is tangent to arc B_aO' at I'.

Ibn Sahl proves this, and the uniqueness of the tangent, reasoning by the absurd.

The plane drawn along line B_cB_d and perpendicular to plane ACI', is tangent to surface (BX) at point I'; this tangent plane is unique.

Ibn Sahl also proves by the absurd that lines AI' and CI' do not meet surface (BX) other than at point I'. The ray from a luminous body along CI' is reflected by mirror (BX) along $I'A$, according to the laws of reflection. This applies to all points on surface (BX).

Figure 10



IV. THE PLANO-CONVEX LENS

In the second part of his treatise, Ibn Sahl is quite naturally led to prove that the hyperbola is an anaclastic curve, and consequently to elaborate the first geometric theory of lenses. This section, which has survived in full, starts with the first stage of the law of refraction.

The hyperbola as a conic section: The law of refraction.

Ibn Sahl first considers refraction on a plane surface. Defining *GF* as the plane surface of a piece of crystal of homogenous transparency, he emphasizes a relation that is the reciprocal of the refractive index *n* of this crystal in relation to air.[19]

> Let *DC* be a light ray in the crystal, which is refracted (Fig. 11; see also Fig. 1) in the air along *CE*. The perpendicular to the plane surface *GF* at *G* intersects line *CD* at *H* and the refracted ray at *E*.

The ratio $CE/CH < 1$, which Ibn Sahl uses throughout his study, is the reciprocal of *n:*

> Let i_1 and i_2 be the angles formed by *CD* and *CE*, respectively, with the normal; we have
>
> $$\frac{CE}{CH} = \frac{CE}{CG} \cdot \frac{CG}{CH} = \frac{\sin i_1}{\sin i_2} = \frac{1}{n}.$$
>
> Let *I* be a point on segment *CH* such that $CI = CE$, and let point *J* be the middle of *IH*. We have $CI/CH = 1/n$. Therefore *C, I, J, H* characterize the crystal for any refraction.

This result of considerable importance, encountered here for the first time, enabled Ibn Sahl to utilize the law of inverse return in the case of refraction, which is essential for the study of biconvex lenses, as we shall see later.

Constructing the lens

To construct a lens allowing burning at a finite distance by parallel rays, Ibn Sahl considers a lens whose substance has the same index of refraction *n* as the crystal just studied.

> Let *A, B, K,* and *L* be points on a straight line, reproducing a division similar to *C, J, I, H*—that is, such that
>
> $$\frac{AK}{AB} = \frac{CI}{CJ} \text{ and } BL = BK;$$
>
> we have therefore
>
> $$\frac{AK}{AL} = \frac{CE}{CH} = \frac{1}{n}.$$
>
> Then let *M* be on *AB* such that $AM = BK$, and place *N* such that *BN* is perpendicular to *AB* and $BN \cdot BM = 4BL \cdot LM$. Consider the hyperbola with vertex *B*, axis *BM*, and *latus rectum BN*.[20] By rotating arc *BS* of this hyperbola about line *AB* we engender a hyperbolic surface; *S* describes a circle with center *O*, and we obtain a revolution solid limited by the hyper-

[19] T, fols. 5–9. Excavation of rock crystal was practiced intensively at Bassorah, according to al-Bīrūnī, *Al-jamāhir fī maᶜrifat al-jawāhir* (Hyderabad: Dāᵓirat al-Maᶜārif al-ᶜOsmānyya, 1936 [1355 A.H.]), p. 184.

[20] This hyperbola, with vertices *B* and *M*, has focuses *A* and *L;* therefore its eccentricity is $e = MB/AL = AK/AL = 1/n$. Therefore the choice of the hyperbola depends on the nature of the crystal.

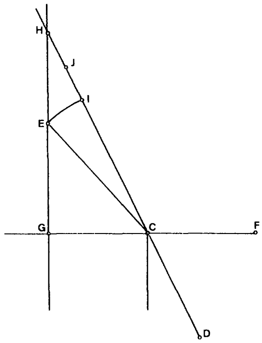


Figure 11

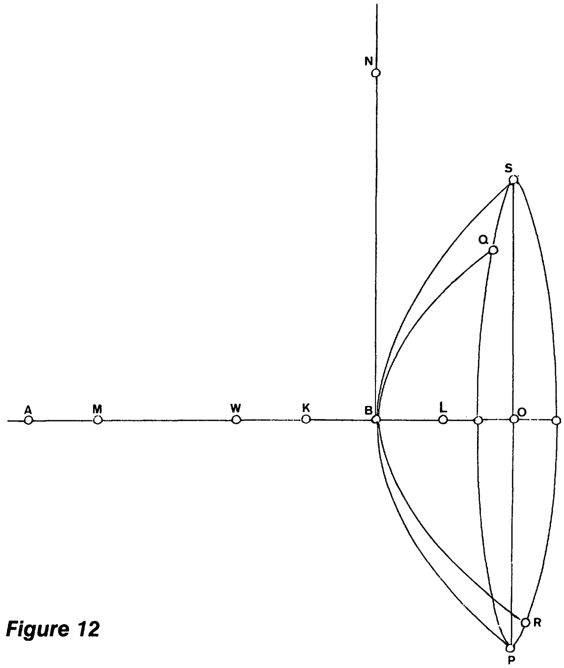


Figure 12

bolic surface and circle (O, OS) (Fig. 12; see also Fig. 1). We suppose this solid to be made of crystal with refractive index n.

PROPOSITION. *Solar rays parallel to* OB *and passing through this solid are refracted at the hyperbolic surface, and the refracted rays converge at* A.

In fact, any ray parallel to OB falling at a point on surface (O, OS) crosses it without refraction and meets the hyperbolic surface either at B or at point $T \neq B$. In the case of point B, Ibn Sahl proves by the absurd that

- the plane perpendicular to OB at B is tangent to the hyperboloid at B;
- the tangent plane at B is unique; and
- line AO meets the hyperboloid only at point B.

He deduces that the ray propagated along OB is perpendicular to the plane tangent at B, does not undergo refraction, and arrives at A.

In the case of point $T \neq B$ (Fig. 13), Ibn Sahl shows that

- plane BLT meets the surface of the lens along the hyperbola VBW of axis BM and focuses A and L;
- the bisector TZ of $\sphericalangle ATL$ is tangent to the hyperbola at T;
- the plane drawn along TZ, perpendicular to plane BLT, is tangent to the hyperbolic surface at T, and this tangent plane is unique.

We have $AT - LT = BM$. Let U' be on TA, such that $AU' = BM$; then $TU' = TL$, TZ is the bisector of LU', and LU' is therefore perpendicular to the tangent plane.

Let XT be the incident ray; XT is parallel to AL, the lines XT, TL, TZ, and TA all being in plane ATL which also contains the normal to the hyperboloid at T; the refracted ray will also be in this plane. The straight line XT cuts LZ at B_a; we have

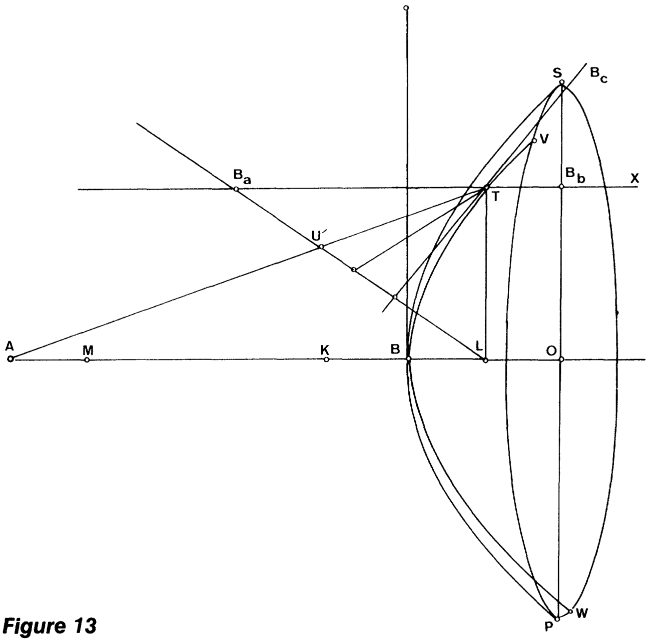


Figure 13

$$\frac{TU'}{TB_a} = \frac{AU'}{AL} = \frac{AK}{AL}.$$

But, by construction,

$$\frac{AK}{AL} = \frac{CE}{CH};$$

therefore

$$\frac{TU'}{TB_a} = \frac{CE}{CH}.$$

Figure TZB_aU' is therefore similar to figure *CGHE; TU'A* is thus the refracted ray corresponding to the incident ray *XT,* which passes through plane *OS* at B_b without deviation and meets the surface of the hyperboloid at *T*.

The bundle of rays parallel to *AB* falling on circle (*O, OS*) penetrates the lens without deviation, and is transformed into a bundle of rays converging at point *A*.

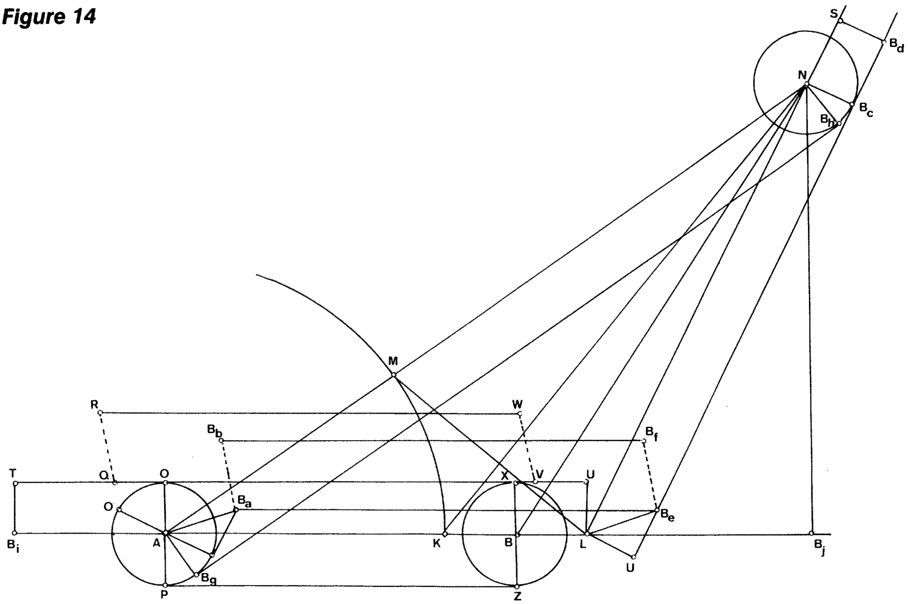
Drawing the hyperbola

To effect the continuous drawing of the hyperbola, Ibn Sahl starts with the division (*A, B, K, L*), which he presented earlier, and therefore obtains $AK/AL = 1/n$, if *n* is the index of the crystal studied.[21]

Let *M* be a point on circle (*A, AK*), such that $\sphericalangle AML$ is obtuse, and place *N* on the straight line *AM,* such that $\sphericalangle MLN = \sphericalangle LMN$; we have therefore $NM = NL$, and $NA - NL = AM = AK$; *N* belongs to the hyperbola with focuses *A* and *L* and with a vertex at *B*.

[21] T, fols. 10–12, 17–19.

Figure 14



As we have seen before, Ibn Sahl does not name the conic section at this stage. He wants to construct arc *BN,* which is none other than a hyperbolic arc. In this case, his approach is inspired by the procedures he has already used for the two other conic sections.

Let *OP* be a segment with bisecting point *A*, perpendicular to *AB*, with $OP \leqslant AB$ and $OP \leqslant KL$ (Fig. 14). On the parallel to *AB* drawn through *O,* place points *U* and *X,* orthogonal projections of points *L* and *B*, respectively, and then place *V* and *Q,* such that $UV = OQ$ (an arbitrary length), and any segment $UT > LN$. Draw circles (*A*, *AO*) and (*B*, *BX*).

On the perpendicular to *LN* at *L,* set U', such that $LU' = LU$, and draw the diameter of circle (*A*) parallel to LU'; label this diameter $O'AI'$. Let $I'B_a$ be perpendicular to AI', with $I'B_a = OQ$; and on the perpendicular to LU' at U', let points B_e, B_c, and B_d be situated such that $U'B_e = OQ$, $U'B_c = LN$, and $U'B_d = UT$. At points *Q, V,* B_a, and B_e raise equal segments perpendicular to plane *ALM:* $QR = VW = B_aB_b = B_eB_f$; therefore we have

$$AL = OU = RW = I'U' = B_aB_e = B_bB_f.$$

If we draw the circle with center *N* equal to circle (*A*), it is tangent to $U'B_c$ at B_c (since $NLU'B_c$ is a rectangle, $NB_c = LU' = AI'$). Draw *PZ* tangent to both (*A*) and (*B*), and B_gB_h tangent to both (*A*) and (*N*); we have $PZ = AB$, $AN = B_gB_h$, $LN = U'B_c$, and $NS = B_cB_d$. We will show that

$$B_gB_h + B_cB_d = PZ + XT. \qquad (1)$$

We have

$$B_gB_h + B_cB_d = AN + NS = AK + MN + NS;$$

but

$$MN + NS = LS = UT = LB_i,$$

B_i being the orthogonal projection of T on AB. We deduce that

$$AN + NS = B_g B_h + B_c B_d = AK + LB_i$$
$$= AK + LB + BB_i = AB + BB_i = l,$$

and for any point on the hyperbola, we have

$$AN + NS = AB + BB_i.$$ [22]

But since $AB = PZ$ and $BB_i = XT$, equality (1) is verified. On the other hand,

$$\overset{\frown}{B_hB_c} = \overset{\frown}{B_gI'} \text{ since } \sphericalangle B_gAI' = \sphericalangle B_hNB_c;$$

therefore $\overset{\frown}{O'PB_g} + \overset{\frown}{B_hB_c} = ½$ circle, and we have

$$\overset{\frown}{O'B_g} + B_gB_h + \overset{\frown}{B_hB_c} + B_cB_d = PZ + \tfrac{1}{2}\text{ circle} + XT = l + p, \quad (2)$$

p being the semiperimeter of one of the circles.

Note that circles (A) and (B) do not intersect, since $AB \geqslant OP$. On the other hand, we have AN > AB, a property of the hyperbola that Ibn Sahl proves by the absurd; therefore $AN \geqslant OP$, and circles (A) and (N) do not intersect.

Now Ibn Sahl starts with equality (2) and imagines a device for making a continuous drawing of the hyperbolic arc BN (Fig. 15). This device consists of a system of two rigid parts. The first part, associated with the fixed point A, pivots about this point; it is formed by the semicircle limited by diameter OP, the segment OQ, and the segment RQ, which is perpendicular to plane LAO. The second part is associated with the fixed point L, about which it pivots, and is formed by the rigid set-square LUT and by segment VW, perpendicular to plane LUT; $VW = QR$ and V is on UT, such that $UV = OQ$. The two parts are joined to each other by shaft RW, functioning as a connecting rod. Any rotation of the second part about L produces the same degree of rotation of the first part about A. To the two rigid parts Ibn Sahl then adds a mobile part consisting of circle (B) functioning as a pulley, and a belt fixed at P and T and turning around (B); its outline $PZXT$ has a constant length $l + p$ according to equality (2).

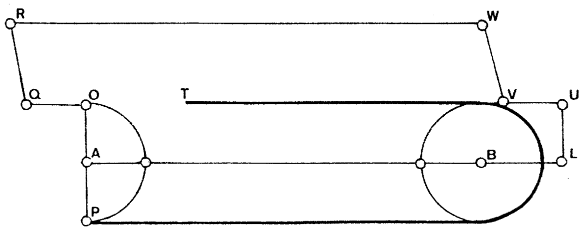
If we push on circle (B), the belt remaining taut, the circle in turn pushes the rigid set-square TUL, which pivots about the fixed point L, pulling the entire rigid device, while shaft RW remains parallel to AL. The set-square LUT will replace $LU'B_d$, P will be at O', and the belt will arrive at position $O'PB_gB_hB_cB_d$, when point B is superposed on point N (Fig. 16). In this displacement, center B of the pulley describes arc BN.[23]

As point M is the intersection of line AN and circle (A, AK), we have $NM < NK$, and hence $NL < NK$. In triangles NBL and NBK, therefore, we have $\sphericalangle LBN < \sphericalangle KBN$, and consequently $\sphericalangle LBN$ is acute. Foot B_j of the perpendicular drawn from N to AB therefore lies on the segment BLB_j. We can then show, by the absurd, that line NB_j does not meet arc BN at any point other than N. By rotation of the figure limited by arc BN and segments BB_j and

[22] Conversely, $AN + NS = AB + BB_i \rightarrow AN + NS - LS = AB + BB_i - LB_i$, whence $AN - NL = AB - BL$.

[23] Note that the movement of the pulley (B) is obstructed neither by shaft RW, which lies above plane ALN, nor by the movement of the set-square LUT, since, according to the condition $OP \leqslant KL$ stated at the beginning of the problem, the radius of the pulley is smaller than distance BL.

Figure 15. *Ibn Sahl's device for drawing hyperbolas, shown at its starting position for drawing a hyperbolic arc. The fine lines indicate the indeformable system; the bold lines, the belt of constant length.*



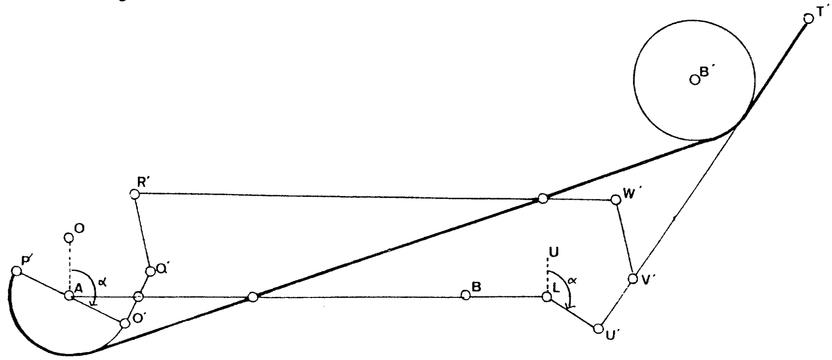


Figure 16. *The drawing device, showing the position reached when both parts of the system have turned angle α, one about point* A, *the other about point* L. *The pulley has moved from position* B *to position* B′.

> NB_j about line BB_j, we engender a solid and suppose it made in the crystal studied earlier.

Refraction at a hyperbolic surface

After completing the continuous drawing of the hyperbola, Ibn Sahl examines the anaclastic property of the curve.[24] He presents the following proposition:

PROPOSITION. *Solar rays parallel to BB_j, falling on surface (B_j), pass through this surface without deviation, and fall on the hyperbolic surface (B); they are then refracted to point A.*

To prove this proposition, Ibn Sahl considers on the hyperbolic surface, successively, point *B*, which is on the axis, and a point other than *B*. He examines the tangent plane and the path of the luminous rays in both cases.

> Consider point *B*. In plane *BNL* we have arc NBB_i, a hyperbolic arc whose vertex is *B* (Fig. 17). Let $BB_{o'}$ be perpendicular to *BL*.

Ibn Sahl shows by the absurd that $BB_{o'}$ is tangent to arc NBB_i at *B*. He also shows by the absurd that no other straight line is tangent to this arc at *B*. He then considers the plane perpendicular to plane *BLN* through line $BB_{o'}$, and shows that it is tangent to surface (*B*) at *B*, and that this is the only tangent plane at this

[24] T, fols. 20–25.

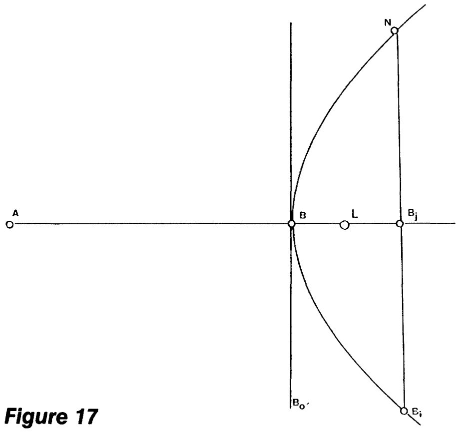


Figure 17

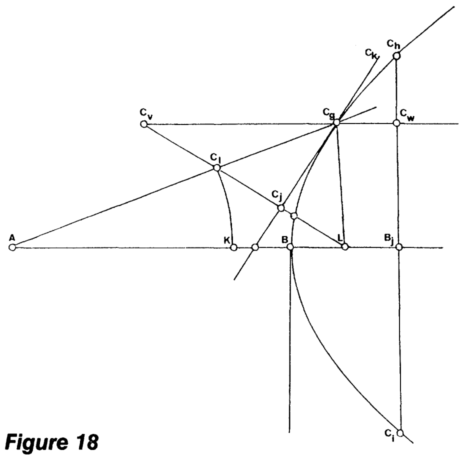


Figure 18

point. Lastly, he shows—by the absurd—that line AL meets surface (B) only at point B.

Thus sunlight is propagated along B_jB in the crystal, and along BA in the air.

Now consider point $C_g \neq B$ (Fig. 18). Plane BLC_g cuts surface (B) along line C_hBC_i.

Ibn Sahl shows by the absurd that the bisector C_gC_j of $\sphericalangle LC_gA$ is tangent to this line at C_g, and that this straight line is the only tangent to this line at C_g. Lastly, he shows that the plane perpendicular to plane ALC_g drawn through line C_gC_j is tangent to surface (B) at point C_g.

Let C_l be the intersection of AC_g and circle (A, AK). Line LC_l meets the tangent at point C_j, and is perpendicular to the tangent plane at C_j. The parallel to AL drawn through C_g intersects plane (B_j) at C_w, and line LC_l at point C_v; we have

$$\frac{C_gC_l}{C_gC_v} = \frac{AC_l}{AL} = \frac{AK}{AL};$$

however, by hypothesis,

$$\frac{AK}{AL} = \frac{CE}{CH},$$

and therefore

$$\frac{C_gC_l}{C_gC_v} = \frac{CE}{CH} = \frac{1}{n}.$$

On the other hand, Ibn Sahl shows by the absurd that C_g is the only meeting point of surface (B) with line C_wC_v and line AC_g.

Consequently, the solar ray parallel to AL, falling on plane (B_j) at C_w, penetrates the crystal and is propagated along C_wC_g; at C_g on surface (B) it is refracted, and it is then propagated in the air along line C_gA.

The same is true for any solar ray falling on surface (B_j).

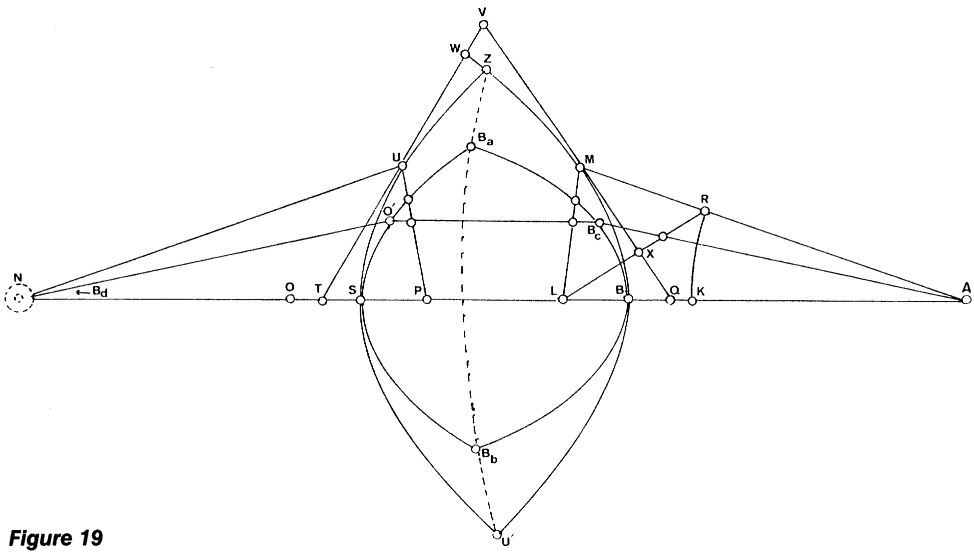


Figure 19

V. THE BICONVEX LENS

Ibn Sahl terminates his study by constructing a lens limited by two portions of a hyperboloid of revolution of the same axis and composed of the same substance as the plano-convex lens, above.[25] In his construction, he uses the result established in the course of his study of that lens, while admitting the principle of the inverse return of light. The biconvex lens constructed here consists of two plano-convex lenses joined together.

As above, Ibn Sahl considers a division *A, K, B, L* on a straight line, similar to division *C, I, J, H,* which is associated with arc *BM* of a hyperbola with vertex *B* and focuses *A* and *L.* He considers a second division *N, O, S, P,* also similar to *C, I, J, H,* associated with a hyperbola of vertex *S* and focuses *P* and *N* (Fig. 19; cf. Fig. 20).

We have

$$\frac{CI}{CH} = \frac{NO}{NP} = \frac{AK}{AL} = \frac{1}{n},$$

n being the refractive index of the crystal in relation to air.

The bisector of $\sphericalangle AML$, labeled *MQ,* is tangent to curve *BM* at *M.* Let *R* be on *AM,* such that $MR = ML$ (therefore $AR = AK$); then *MQ* meets *LR* at *X* in a right angle, and therefore $\sphericalangle LQX$ is acute. Similarly, the bisector of *NUP,* labeled *UT,* is tangent to curve *SU,* and $\sphericalangle PTU$ is acute. Therefore lines *MQ* and *TU* meet each other; let this be at point *V.*

Curve *BM* meets line *QB* only at point *B;* it meets line *QM* only at point *M,* and it meets line *TV* at only one point, labeled *W.* Curve *SU* meets line *TV* only at point *U;* it meets curve *BW* at point *Z.*

If we fix line *BS* and pivot the surface limited by arcs *BZ* and *ZS* and line *BS* about it, point *Z* describing circle *ZU',* we obtain the solid *BZSU'.* We make a solid equal to this one from the same crystal previously considered.

[25] T, fols. 25–26.

PROPOSITION. *Luminous rays proceeding from N, falling on surface ZSU′, penetrate this solid, meet surface ZBU′, and are propagated to point A; they burn at this point.*

Start by considering the case for point S. Line NS meets the surface of the luminous body at I'. Ray $I'S$ penetrates the solid at S, is propagated along SB, emerges at B, and is propagated along BA.

Then consider any point $O' \neq S$. Plane BSO' intersects the surface of the solid along $SO'B_a$ and B_aB_cB (B_a is a position of Z, arc $SO'B_a$ a position of arc SZ, and arc B_aB_cB a position of arc ZB); we suppose $O'B_c$ parallel to BS. Line NO' meets the surface of the luminous body at point B_d. The light at point B_d is propagated in the air along B_dO', penetrates the solid at O', is propagated along $O'B_c$, emerges at B_c, and is propagated along B_cA.

A bundle of rays emanating from point N undergoes a first refraction at surface S that transforms it into a cylindrical bundle; when this bundle falls on surface B, it is refracted a second time and is transformed into a bundle of rays that will converge at point A.

VI. IBN SAHL'S CONTINUOUS DRAWING OF CONICS: THE PERFECT COMPASS

Tenth-century mathematicians paid particular attention to the construction of conics by continuous drawing. For that purpose they devised certain instruments, including the perfect compass, and several wrote treatises on them. Among those, who composed such a treatise was al-Qūhī, whose works were known to Ibn Sahl, since the latter commented on one of them, a *Treatise on the Art of the Astrolabe*. Al-Qūhī affirmed that he knew of no work on the perfect compass by Greek geometers.[26] Ibn al-Haytham also wrote such a text, for he refers to it in his treatise on parabolic burning mirrors, as one in which he showed "the determination of all conic sections by the instrumental method: how to determine conic sections so accurately [*ᶜalā ḥaqīqatihi*] that no other more accurate may be achieved on the matter, as in determining the existence of a circle by the compass." This latter treatise is still undiscovered.[27] Although some of these compasses have already been studied, an analysis of their interconnections has yet to be made. I shall now consider the instruments devised by Ibn Sahl and endeavor to extract from their apparent complexity the idea on which they are based. I shall then give a brief summary of the principle of the perfect compass.

Ibn Sahl's devices for the continuous drawing of the three conic sections each

[26] See Franz Woepcke, "Trois traités arabes sur le compas parfait," *Notices et extraits des manuscrits de la Bibliothèque Impériale et autres bibliothèques*, n.d., Vol. XXII, Part 1, pp. 68, 145. For al-Qūhī's treatise on the astrolabe and Ibn Sahl's commentary see Rashed, *Géométrie et dioptrique* (cit. n. 2).

[27] Ibn al-Haytham, *Fī al-marāyā al-muḥriqa bi al-quṭūᶜ* (On burning mirrors by conic sections), p. 11; translated into Latin by Gerard of Cremona as *Liber de speculis comburentibus*. The Latin text, with a German translation of the Arabic text, was published by Johan Ludvig Heiberg and Eilhard Wiedemann, "Ibn al-Haitams Schrift über parabolische Hohlspiegel," *Bibl. Math.*, 3rd Ser., 1910/11, *10*:193–208. On the role of the text for our understanding of conics in Latin science see Marshall Clagett, *Archimedes in the Middle Ages* (Philadelphia: American Philosophical Society, 1980), Vol. IV, pp. 13–18. An English translation also exists by H. J. Winter and Walid ᶜArafāt: "Ibn al-Haitham on the Paraboloidal Focusing Mirrors," *Journal of the Royal Asiatic Society of Bengal*, 3rd Ser. (Science), 1949, *15*:25–40.

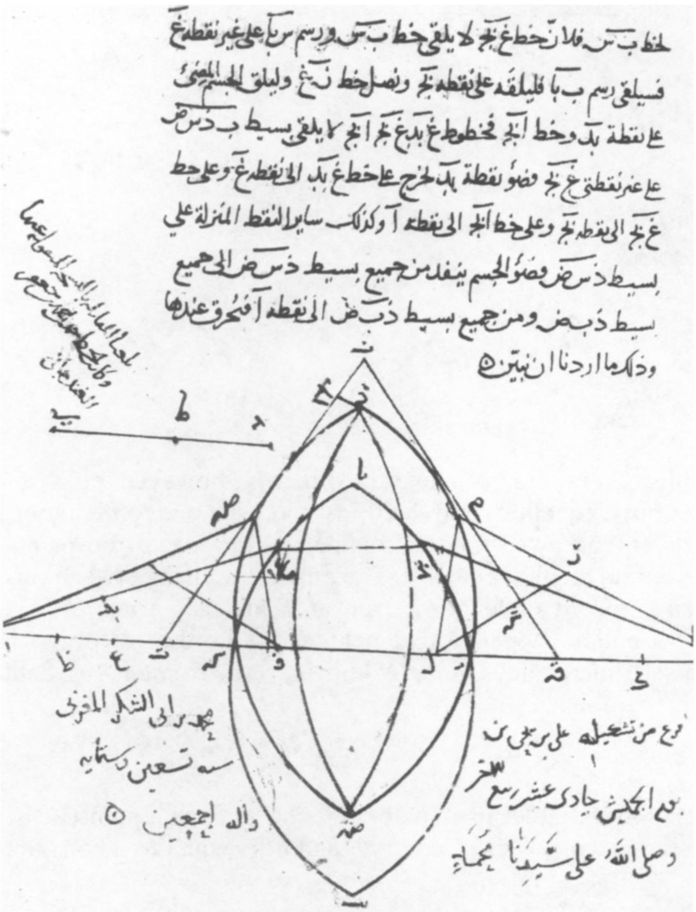


Figure 20. *The biconvex lens as depicted in Millī MS 867, folio 26r; compare Figure 19. Courtesy of Millī Library, Tehran.*

have, as we have seen, two components: one rigid, the other flexible though of constant length (see Figs. 6, 14, 15, and 16). The flexible component consists of a cord or belt that turns about a mobile circle, acting as a pulley, whose function is to prevent the breaking of the cord while facilitating the movement of the mobile part. If the center of this circle is equipped with a stylus it will draw the arc of the curve studied.

For each of the three conic sections, which we will consider in turn, it is a property of the focus that is important.

The parabola

Let there be a parabola with focus F, and a straight line Δ perpendicular to the axis that cuts the parabola at points A and B (Fig. 21). For every point M on arc AB with projection H on Δ, we have

$$MF + MH = l, \text{ and } AF = BF = l, \tag{1}$$

l being the distance from Δ to the directrix D. We have, on the other hand,

$$MF = MH'. \tag{2}$$

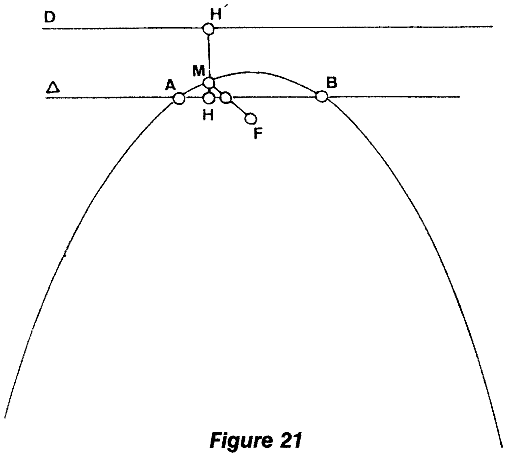


Figure 21

Ibn Sahl, like his predecessors, did not name the directrix; however, his demonstration uses the two above equalities and the transition from one to the other.

If we examine the device conceived for the continuous drawing of the parabola, we note that it is based on the first equality. The only difference between this device and the one using a cord of length l fixed at F and at vertex H of a set-square is the use of a pulley. A pencil that tightens the cord at M draws a parabolic arc when the set-square slides along Δ: this is the instrument Ibn Sahl imagined for drawing the parabola.

The ellipse

Ibn Sahl uses the property that enables us to determine the locus of a point M the sum of whose distances between two fixed points F and F' is equal to a constant magnitude l—that is,

$$MF + MF' = l;$$

F and F' are the two focuses of the ellipse, and l is the length of the major axis. The device that Ibn Sahl proposes differs from the well-known "gardener's method" only in the use of pulleys, of which two are fixed and one is mobile (see Fig. 22).

The hyperbola

> Let there be a hyperbola with focuses F and F', whose transverse axis has a length $2a$. Any point M on the branch that surrounds point F is characterized by
>
> $$MF' - MF = 2a.$$
>
> Let S be a point on the prolongation of FM (Fig. 23); we have
>
> $$(SM + MF') - SF = 2a.$$

Given these relations, it is possible to imagine a device for the continuous drawing of a hyperbolic arc:

> Consider a ruler that pivots about focus F, and a cord attached at one end to focus F' and at the other end to point S on the ruler. If the distance

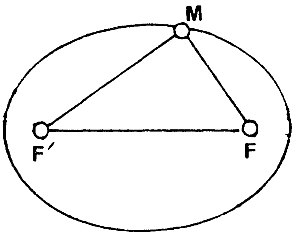


Figure 22

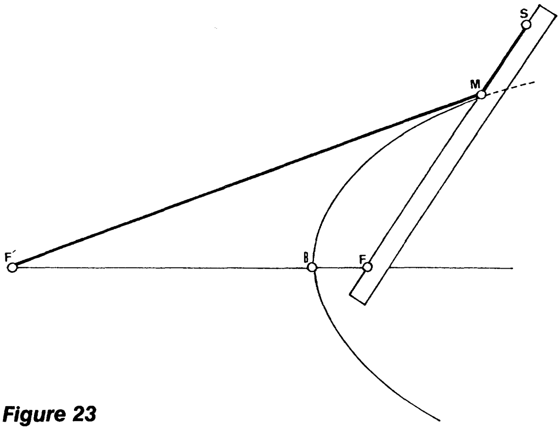


Figure 23

between the two points F and S on the ruler is $FS = l$, we take a piece of cord of length $l' = l + 2a$. We keep the cord taut with the help of a pencil resting at M on the ruler: the pencil point draws arc MB when the ruler pivots about F.

Now let us turn to the device imagined by Ibn Sahl for drawing the hyperbola, which evolved precisely from the idea we have just set forth. He uses two pulleys of the same radius—one with a fixed center, the other with a mobile center—on which the cord or belt of invariable length rests.

Ibn Sahl must have been acquainted with existing contemporary works on the perfect compass—I referred earlier to his commentary on the *Treatise on the Art of the Astrolabe* by al-Qūhī, author of a treatise on the perfect compass. Al-Qūhī's instrument consists of three articulated parts (Fig. 24): part MN, called the base of the compass, corresponds to axis V of the conic; part LP, called the axis of the compass, corresponds to the axis of the cone. Line segment RQP, called the drawing pen, revolves about line PL; its length is variable, which enables pen point R to remain in contact with plane π during rotation, and thus to draw the conic section.

The perfect compass therefore draws a conic section if we know, for example, the *latus rectum*, a diameter, and the angle formed by this diameter and the conjugate direction. However, this drawing requires preliminary constructions to

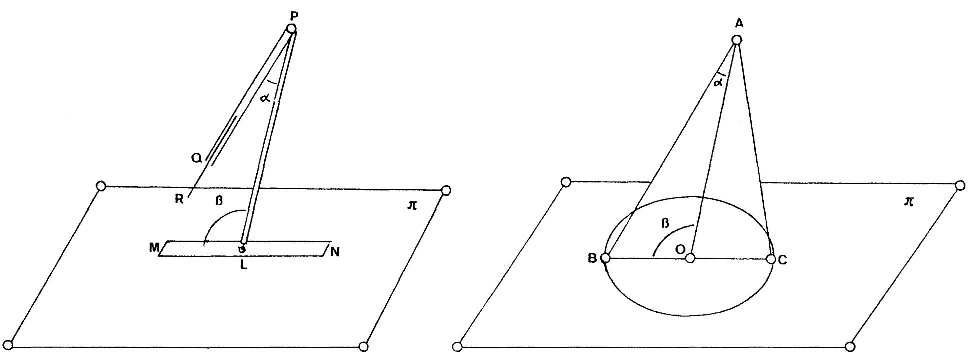


***Figure 24.** Al-Qūhī's perfect compass and the corresponding cone.*

determine the two angles, α and β, of the perfect compass (which are equal in the case of the parabola). Ibn Sahl may indeed have proposed his method in order to dispense with such lengthy and complicated preliminary constructions. This conjecture seems plausible, but here as elsewhere Ibn Sahl does not reveal his intentions.

As to whether the method that Ibn Sahl used to construct conic sections survived, I suggest that it did, on the following evidence. As already noted, Ibn Sahl's successor Ibn al-Haytham, in his work on the parabolic mirror, mentioned a treatise he had composed on the construction of conic sections "using the instrumental method" (*bi-ṭarīq al-ᵓāla*). He wrote: "As for the way to determine the parabola and other conic sections using the instrumental method, this was mentioned by a group of geometers, even if they did not determine it exactly [*ᶜalā ḥaqīqatihi*]";[28] this last phrase suggests that Ibn al-Haytham had perfected the method himself. I suspect that among this "group of geometers," he had been thinking first and foremost of Ibn Sahl.

VII. CONCLUDING REMARKS

As we have seen, it was the study of burning mirrors that led Ibn Sahl to the first investigations on anaclastics; and his mastery of the theory of conics, as revealed not only in this treatise but also in his mathematical work analyzed elsewhere, made the birth of this chapter possible. Just as in research on burning mirrors, in anaclastics one proceeds by applying geometric structures, notably supplied by the theory of conics, to certain luminous phenomena so as to achieve the practical goal aimed at from the beginning: burning. The constructed object—whether mirror or lens—must therefore conform to the geometric structure applied. Some would say today, and justifiably so, that it is a matter of constructing models with the aid of mathematics to achieve a practical end: that of burning from a distant or near source. In this respect, the only differences between the new chapter on anaclastics and the much older one on burning mirrors are the complexity of the phenomena studied and the subtlety of the mathematical structures employed.

Consequently, renewed interest in the study of refraction in the tenth century is fully justified: for the first time since Ptolemy's *Optics*, real progress was made in the knowledge of refraction. Ibn Sahl, who read and commented on the Alexandrian writer, knew that the incident ray and the refracted ray lie on a plane containing the normal, on either side of it. Similarly, he knew the principle of inverse return. To this he added the ratio he discovered, equivalent to Snellius's law as the latter would later formulate it: as noted, he introduced the ratio of the refracted ray to the distance of the image at the point of incidence (CE/CH throughout his study) as a constant ratio for two given media. Even if he did not state the law explicitly, it underlies all of his research on lenses, and his contribution is of the utmost importance. (Ibn Sahl's discretion in this matter is apparently not fortuitous: it would seem due to the absence of inquiry into the physical causes of refraction—that is to say, to the lack of any attempt to justify this mode of propagating light.)

The discovery of this treatise by Ibn Sahl compels us to consider the connec-

[28] Ibn al-Haytham, *Majmūᶜ al-Rasāᵓil* (Hyderabad: Dāᵓirat al-Maᶜārif al-ᶜOsmānyya, 1938 [1357 A.H.]), p. 11.

tions between Ibn al-Haytham and his predecessor, and thus it enables us to place more exactly in its historical context a contribution that historians readily qualify as revolutionary. If we confine ourselves to the study of lenses alone, we observe that Ibn Sahl considers only rays parallel to the axis and obtains the convergence of all refracted rays at a single point on the axis. Let us remark that, for Ibn Sahl, the notion of focus of a conic ceases to be connected simply with reflection; it is henceforth linked to refraction as well.

Nonetheless, it remains true that Ibn Sahl's sole intention was burning and his study is purely geometric. At no time does any kind of experimentation whatsoever intervene as part of his proof. Because he only wanted to burn, he confined himself to the conception and construction of a geometric model that would help to construct the mold of the lens. As a result he refined and advanced geometric study; the practical value and effectiveness of the model were to be tested when the model was eventually used. But when attention is paid to the problems raised by the image of an object observed through the lens, the situation becomes quite different; in this case it is impossible to avoid difficulties such as astigmatism and aberration. Such problems, unimagined in Ibn Sahl's treatise, would arise in the work of Ibn al-Haytham, who would be led to redefine the relationships between conditions for vision and conditions of light and its propagation, between sight and illumination.

It would have been surprising for such a major contribution to the history of optics, remarkable for its time, to remain without an heir. And it would be equally surprising if earlier important works had not paved the way for a work as revolutionary as that of Ibn al-Haytham. We know that Ibn al-Haytham was in fact acquainted with Ibn Sahl's writings and with the treatise reconstructed here. His achievement was to consolidate this chapter on anaclastics while expanding its scope—but that is the topic of another study.[29] For the time being, let me simply stress that through our recently acquired knowledge of Ibn Sahl's contribution, we are now in a position to assess more precisely Ibn al-Haytham's contribution to anaclastics and his work on optics in general.

In conclusion, let me note the insufficiently emphasized impact of the study of optical instruments—mirrors and lenses—on the interest in geometric constructions in the tenth and eleventh centuries. The search for mechanical means of constructing conic sections was related to research in optics, just as the construction of the perfect compass during the same period was related to research in astronomy, and in particular to the construction of astrolabes and sundials.

[29] See Rashed, *Géométrie et dioptrique* (cit. n. 2).